Qualifying Exam in Mathematics

May 2007

(11) 1. Consider the initial value problem

$$y''(t) + 2y'(t) + 2 = e^{-t} + \delta(t-3), \qquad y(0) = 0, \ \ y'(0) = 0. \tag{1}$$

(a) Explain briefly the physical meaning of the delta function $\delta(t-3)$ for equation (1).

(b) Find the Laplace transfrom $Y(s) = \mathcal{L}(y)(s)$ of the solution y to (1).

(c) Invert $Y(s)$ found in (b) to obtain the solution $y(t)$ to (1). (You may leave convolutions in your answer, but write out any convolution as an integral. However, your answer should not contain an integral in which $\delta(t-a)$ appears in the integrand; such an expression should be evaluated so that it does not contain an integral.)

(15) 2. (a) Using separation of variables, find the *general* solution to

$$\begin{gathered} u_t = u_{xx}, \quad 0 < x < 1 \\ u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0. \end{gathered}$$

(b) Find a particular solution to

$$\begin{gathered} u_t = u_{xx} + 1, \quad 0 < x < 1 \\ u_x(0,t) = 1, \quad u(1,t) = 0, \quad t > 0. \end{gathered}$$

(c) Find the solution to

$$\begin{gathered} u_t = u_{xx} + 1, \quad 0 < x < 1 \\ u_x(0,t) = 1, \quad u(1,t) = 0, \quad t > 0, \\ u(x,0) = 3. \end{gathered}$$

Your solution should have the form of an infinite series. You should give a formula for the coefficients of this series but you do not need to find explicit expressions for the coefficients.

(12) 3. Consider the equation

$$x^2y'' + 4xy' + (2+x)y = 0. \tag{2}$$

(a) Explain why $x_0 = 0$ is a regular singular point of this equation.

(b) Find the indicial equation and verify that $r = -1$ is a root of this equation.

(c) Consider finding the Frobenius solution $\sum_{n=0}^{\infty} a_n x^{n+r}$ for $r = -1$. Write down the recurrence relation for the coefficients a_n. Find the first three non-zero terms of this solution explicitly, assuming $a_0 = 1$.

(d) Does (2) necessarily have a solution of the form $\sum_{n=0}^{\infty} b_n x^{n+r_2}$, where $r_2 \neq -1$ is the other root of the indicial equation? If not, what form might this second solution take? (It is not necessary to compute any terms of this second solution; just give its general form.)

(12) 4. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y(1) = 0.$$

(a) Determine if $\lambda = 0$ is an eigenvalue. If it is, find an associated eigenfunction.

(b) Find an equation that $\lambda > 0$ must satisfy to be an eigenvalue. If λ_n is the n^{th} eigenvalue, find a_n so that $a_n \cos(\sqrt{\lambda_n} x) + \sin(\sqrt{\lambda_n} x)$ is an eigenfunction associated to λ_n.

(c) Using the equation for positive eigenvalues derived in (b), show graphically that the smallest positive eigenvalue λ_1 satisfies $\lambda_1 < \pi^2$. Also, explain graphically why there are an infinite number of positive eigenvalues.

(d) Let λ_n and ϕ_n denote the eigenvalues and eigenfunctions for the Sturm-Liouville problem (1). Suppose you want to express the function x on the interval $0 \le x \le 1$ as $\sum_1^\infty a_n \phi_n(x)$. Give a formula for a_n. It is not necessary to compute a_n explicitly.

(10) 5. (a) Find all cube roots of $8i$, and give a plot showing the location of each these roots in the complex plane.

(b) Let C be a contour which is the boundary of the square with corners 2, -2, $-2-4i$, and $2-4i$, traced counterclockwise. Evaluate

$$\oint_C \frac{dz}{(z^3 - 8i)(z^2+9)}.$$

(12) 6. Let

$$f(z) = \frac{1}{1 - \cos z}.$$

(a) Find the first three nonzero terms of the Laurent series of $f(z)$, in powers of z, which converges for $|z|$ small.

(b) Determine the region in which the Laurent series that you found in (a) will converge.

(c) What will be the radius of convergence of the Taylor series, with center $z = 1$, of the function $g(z) = z^2 f(z)$? Do not attempt to find this series explicitly.

(14) 7. Let D be the region in the complex plane which is inside the unit circle and lies in the upper half plane, that is, the region in which $|z| < 1$ and, with $z = x + iy$, $y > 0$.

(a) Find a conformal mapping of D onto (i) the upper half plane, and (ii) the strip $0 < \operatorname{Im} z < 1$. In each case, justify carefully that the mapping you provide has the correct image.

(b) Using (a), find a bounded harmonic function $u(x, y)$ defined in D and taking boundary values $u = 0$ on the semicirular part of the boundary of D and $u = 1$ on the portion of the real axis bounding D.

(14) 8. Use the residue theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+4)^2}\, dx\ .$$

Show clearly, and justify carefully, each step,

Laplace Transforms

In the table below, $F(s)$ is the Laplace transform of $f(t)$, $G(s)$ is the Laplace transform of $g(t)$, and H is the Heaviside function, and $f^{(n)}$ is the n^{th} derivative of f.

e^{at}	$\frac{1}{s-a}$ $s > a$
$t^n e^{at}$ (n = non-negative integer)	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$H(t-a)$	$\frac{e^{-as}}{s}$, $s > 0$
$H(t-a)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$f * g(t)$	$F(s)G(s)$
$\delta(t-a)$	e^{-as}

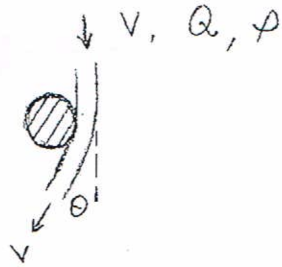
PhD Qualifying Exam

Fluid Mechanics (180 min.)

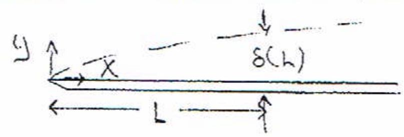
May 2007

Closed books, closed notes. You may use one 8 1/2" × 11" sheet of notes during the exam. Please write your name on the note sheet and turn it in with your exam. **Work neatly and clearly.** For each problem, list assumptions, show work, and explain your reasoning carefully!

1. A circular cylinder inserted across a stream of fluid flowing at speed V and volume flow rate Q deflects the stream through angle θ (this is called the Coanda effect). Find the direction and magnitude of the horizontal component of the force on the cylinder caused by the flowing fluid. Neglect gravity.



2. Consider a thin, incompressible, steady boundary layer developing on a long flat plate. Show by non-dimensionalizing the governing equations (or in some other way),



   (a) Pressure across the boundary layer is constant ($\frac{\partial P}{\partial y} \simeq 0$).

   (b) $\frac{\delta}{L} \propto Re_L^{-1/2}$.

3. Consider the following form of a steady thermal-energy equation:

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} , \qquad (1)$$

   where T is the temperature and k is the thermal conductivity.

   (a) Use the Reynolds decomposition and averaging to derive the mean thermal-energy equation for $\overline{T}$.

   (b) Rewrite the mean thermal energy derived in part (a) in the following form:

$$\rho c_p \left(u \frac{\partial \overline{T}}{\partial x} + v \frac{\partial \overline{T}}{\partial y} \right) = -\frac{\partial}{\partial y} (q^{\text{molecular}} + q^{\text{turb}}) \qquad (2)$$

   and identify the turbulent contribution to the heat flux q^{turb}, where, $q^{\text{molecular}} = -k \frac{\partial \overline{T}}{\partial y}$.

   (c) Consider a heated plate in turbulent flow. State the expected sign (positive or negative) of the turbulent contribution to the heat flux, and explain why.

4. Consider inviscid, irrotational supersonic flow over a sinusoidal surface defined by

$$y = a \cos kx \, , \tag{3}$$

where a is the amplitude and k is the wavenumber and $ka << 1$. The velocity potential is

$$\Phi = U_\infty x + \phi(x,y) \, , \tag{4}$$

where U_∞ is the mean velocity and $\phi(x,y)$ is the perturbation velocity potential satisfying

$$\left(M_\infty^2 - 1\right) \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \, . \tag{5}$$

(a) Find the perturbation velocity potential $\phi(x,y)$ for $-\infty < x < \infty$ and $y > 0$.

(b) Find the surface pressure coefficient c_p defined by

$$c_p = \frac{(p - p_\infty)}{\frac{1}{2}\rho_\infty U_\infty^2} \, . \tag{6}$$

Rutgers University
Mechanical and Aerospace Engineering

Ph.D. Qualifying Exam 2007
Thermodynamics (90 min.)

Thermodynamics

Part A. Solve both problems below.

(1) An infinitely long tube with radius a = 0.01 m is filled with gas containing only one molecular species. The gas is in thermal equilibrium with a number density n = 2.6 x 10^{25} molecules/m^3, temperature T = 300 K, and collisional of cross section $\sigma = 10^{-19}$ m^2. Determine the ratio of the number of collisions between molecules in the gas to collisions with the wall.

(2) An inert gas thruster used in satellite station-keeping is simply a chamber with a small hole (of diameter d) which uses exhaust of the propellant into vacuum. In the thruster design, the mean free path of the molecules in the chamber is $\lambda >> d$.

(a) Write down an expression for the velocity distribution function of the particles exhausted from the chamber through the hole, and sketch the distribution function for the component of the velocity normal to the plane of the hole.
(b) Derive the expression for the thrust, T = P*A, where P* is the momentum flux of particles exhausted through the hole and A is the area of the circular hole.
(c) If, for a collection of molecules belonging to a specific velocity class, the entropy per molecule, $s(C_i) = k - k\ln\left[\frac{h^3}{m^3} nf(C_i)\right]$, then calculate the entropy flux, Λ_s.

Part B. Solve 1 of the 2 problems below.

(1) For the elementary reaction, $H + Cl_2 \rightarrow HCl + Cl$, the forward rate constant, k_f = 4×10^{13} cm^3/mole-s independent of temperature, and the reverse rate constant, k_r, is given by 1 x 10^{13} exp(-45,400/$\check{R}$T) cm^3/mole-s. $\check{R}$ is 1.987 cal/mole-K.
(a) What is the heat of reaction, ε_0, or ΔHr, in kcal/mole?
(b) Is the forward reaction exothermic or endothermic?
(c) If, at equilibrium at 1000K, the number densities of Cl_2 and HCl happen to be the same, what is the ratio of partial pressures of H and Cl?

(2) Evaluate the stability of a simple single-component thermodynamic system, which obeys the fundamental relation,

$$S = C\,[NVU]^{1/3},$$

where C is a positive constant.

Ph.D. Qualifying Examination 2007
HEAT CONDUCTION (90 min)

Do all problems.

Problem 1. Obtain the temperature profile in a kidney tissue slab of 20 mm thick and calculate the maximum temperature inside the tissue. The metabolic rate in the tissue is 10^4 W/m^3. The thermal conductivity of kidney tissue is 0.50 Wm^{-1}K^{-1}. Assume that the temperature at the kidney boundary is 36 °C and 80% of the energy consumed to maintain the metabolic processes is converted to heat. Neglect blood perfusion.

Problem 2. (i) Using a scaling analysis of the transient heat conduction in a semi-infinite medium, estimate the thermal penetration depth δ as a function of time. The initial temperature of the medium is T_i and the temperature of the surface at times $t > 0$ is imposed to be T_s which is greater than T_i. (ii) Develop an integral equation for this problem. (iii) Define an appropriate similarity variable that will convert the equation to an ODE. (Don't solve the ODE unless you have enough time).

Problem 3. Consider the heat conduction equation given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g(r,z) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Derive the finite-difference form of this equation for an internal node (i,j) using the explicit method. What is the stability criterion? (Let $T_{i,j}^n = T(i\Delta r, j\Delta z, n\Delta t)$).